Lecture Note on Linear Algebra
9. Matrix Partition

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1 What Do You Learn from This Note

In this lecture, we shall introduce a powerful technique on manipulating matrices — matrix partition (矩阵分块).

Basic concept: matrix partition (矩阵分块)

2 Matrix Partition

We first see an example about the partition of a matrix.
Example:
The matrix $A = \begin{pmatrix} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ -8 & -6 & 3 & 1 & 7 & -4 \end{pmatrix}$ can be rewritten as:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \tag{1}$$

where

$$A_{11} = \begin{pmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{pmatrix},\ A_{12} = \begin{pmatrix} 5 & 9 \\ 0 & -3 \end{pmatrix},\ A_{13} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} -8 & -6 & 3 \end{pmatrix},\ A_{22} = \begin{pmatrix} 1 & 7 \end{pmatrix},\ A_{23} = \begin{pmatrix} -4 \end{pmatrix},$$

Generally, for a matrix A,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

, the matrix partition means the matrix A is divided into blocks as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{s1} & A_{s2} & \cdots & A_{sk} \end{pmatrix}$$

where A_{ij} is a $m_i \times n_j$ matrix, and $m_1 + m_2 + \cdots + m_s = m$, $n_1 + n_2 + \cdots + n_s = n$. We also call A_{ij} is the block matrix (sub-matrix) of matrix A.

3 Some Properties

Given two matrices A and B and their matrix partition in the following forms:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{s1} & A_{s2} & \cdots & A_{sk} \end{pmatrix}, \; B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1l} \\ B_{21} & B_{22} & \cdots & B_{2l} \\ \cdots & \cdots & \cdots & \cdots \\ B_{t1} & B_{t2} & \cdots & B_{tl} \end{pmatrix}$$

Addition & Scalar Multiplication: If (1) A and B are with the same size and (2) they are partitioned in exactly the same way (i.e. $s = t$, $k = l$, and A_{ij} and B_{ij} are with the same size for any i and j), then we have

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1l} + B_{1l} \\ A_{21} + B_{21} & A_{22} + B_{22} & \cdots & A_{2l} + B_{2l} \\ \cdots & \cdots & \cdots & \cdots \\ A_{t1} + B_{t1} & A_{t2} + B_{t2} & \cdots & A_{tl} + +B_{tl} \end{pmatrix} \tag{2}$$

and

$$r \cdot A = \begin{pmatrix} r \cdot A_{11} & r \cdot A_{12} & \cdots & r \cdot A_{1k} \\ r \cdot A_{21} & r \cdot A_{22} & \cdots & r \cdot A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ r \cdot A_{s1} & r \cdot A_{s2} & \cdots & r \cdot A_{sk} \end{pmatrix} \tag{3}$$

Multiplications of Partitioned Matrices

We say that the partitions of A and B are conformable (相一致) for block multiplication if

1. The column partition of A matches the row partition B, i.e. $k = t$

2. The width of block matrix A_{ij} is the same as the height of block matrix B_{jp}, for any $i = 1, \cdots, s$ and $p = 1, \cdots, t$.

Example: See textbook P135 (板书)

If two matrices A and B are conformable, then AB can be expressed as follows:

$$AB = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1l} \\ C_{21} & C_{22} & \cdots & C_{2l} \\ \cdots & \cdots & \cdots & \cdots \\ C_{s1} & C_{s2} & \cdots & C_{sl} \end{pmatrix} \tag{4}$$

where

$$C_{ip} = A_{i1}B_{1p} + A_{i2}B_{2p} + \cdots + A_{ik}B_{kp} = \sum_{q=1}^{s} A_{iq}B_{qp}. \tag{5}$$

THEOREM 1. ***Column-Row Expansion of*** *AB: If A is $m \times n$ and B is $n \times p$, then*

$$AB = [col_1(A)\ col_2(A)\ \cdots\ col_n(A)] \begin{bmatrix} row_1(B) \\ row_2(B) \\ \cdots \\ row_n(B) \end{bmatrix} = col_1(A)row_1(B) + \cdots + col_n(A)row_n(B) \tag{6}$$

Inverse of Partitioned Matrices: Example on Page 137 (Left for your reading)

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