Lecture Note on Linear Algebra

11. Determinants

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1 What Do You Learn from This Note

Recall the theorem 4 on page 119: For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. So what is $ad - bc$? How to compute $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

In fact $ad - bc$ is called determinant and the matrix is computed by cofactor computation. Introducing them is the objective of this lecture note.

Basic concept：determinants (行列式), cofactor (余子式/余因子)

2 Determinants

Our study is on the following matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{m2} & \cdots & a_{nn} \end{pmatrix} \tag{1}$$

Denote $\overline{A}_{ij}$ be the rest submatrix formed by deleting the i^{th} row and j^{th} column of A. (注意：课本用A_{ij}表示，但这与前面矩阵分块A_{ij}的符号的含义不同，所以讲义中用$\overline{A}_{ij}$表示)

DEFINITION 1 (determinant (行列式)). A determinant of an $n \times n$ square matrix $A = [a_{ij}]$ is defined as follows:

$$\det A = a_{11} \det \overline{A}_{11} - a_{12} \det \overline{A}_{12} + \cdots + (-1)^{1+n} a_{1n} \det \overline{A}_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \overline{A}_{1j}. \tag{2}$$

Always, we also denote the determinant of matrix A by

$$|A| \tag{3}$$

注意：只有方阵才有行列式的定义。

DEFINITION 2 (cofactor (余子式/余因子)). $C_{ij} = (-1)^{i+j} \det \overline{A}_{ij}$ *is called the* (i, j)*-cofactor of matrix* A*. We then call* $\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \overline{A}_{1j}$ *as a cofactor expansion (余子式/余因子展开式) across the first row of* A*.*

In fact, we can compute the determinant of a matrix in a more flexible way:

THEOREM 3. $\det A = a_{i1} \det \overline{A}_{i1} - a_{i2} \det \overline{A}_{i2} + \cdots + (-1)^{i+n} a_{in} \det \overline{A}_{in} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \overline{A}_{ij}$.

Also, we can compute the determinant by the cofactor expansion down the jth column of matrix A as follows:

THEOREM 4. $\det A = a_{1j} \det \overline{A}_{1j} - a_{2j} \det \overline{A}_{2j} + \cdots + (-1)^{n+j} a_{nj} \det \overline{A}_{nj} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det \overline{A}_{ij}$.

注：以上证明本课程及教学大纲不作要求。具体证明可参见：《高等代数与解析几何》（上册），孟道骥著，91页定理一。

Examples: For $n = 3$, the determinant of matrix A is

$$\det A = a_{11} \det \overline{A}_{11} - a_{12} \det \overline{A}_{12} + a_{13} \det \overline{A}_{13}. \tag{4}$$

So when $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$, $|A| =$?(板书)

THEOREM 5. *If* A *is a triangular matrix, then* $\det A$ *is the product of the entries on the main diagonal of* A*.*

3 Properties of Determinant

3.1 Determinant on Matrix Transpose

THEOREM 6. *If A is an $n \times n$ matrix, then* $\det A^T = \det A$

Proof. We prove this theorem using induction method.

STEP 1: for $n = 2$ (e.g. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$), it is true that $\det A^T = \det A = ad - bc$.

STEP 2: for $n > 2$, the cofactor expansion across the first row of A^T is

$$\det A^T = \sum_{j=1}^{n} (-1)^{1+j} [A^T]_{1j} \det \overline{A^T}_{1j}$$

. Note that $[A^T]_{1j} = [A]_{j1}$, and $\det \overline{A^T}_{1j} = \det \overline{A}_{j1}^T = \det \overline{A}_{j1}$ since $\overline{A}_{j1}^T$ is a $(n-1) \times (n-1)$ matrix (因此根据归纳法，我们已经假设对于$(n-1) \times (n-1)$矩阵定理成立). Therefore, we have

$$\det A^T = \sum_{j=1}^{n} (-1)^{1+j} [A]_{j1} \det \overline{A}_{j1} = \det A$$

(注意，我们这里用到了定理3和4，即计算一个矩阵的行列式可以在列和行不同方向展开). □

Remark: Since the i–th row of A is the transpose of the i–th column of A. $\det A^T = \det A$ indicates that any properties and results of $\det A$ relating to columns of A also hold for rows of A.

注：以上定理也表明对矩阵A^T做行变换等价于对矩阵A做列变换.我们在课程中没有仔细介绍列变换，但实际上等价于对其转置矩阵的行变换。

3.2 Determinants of Elementary Matrix

THEOREM 7.
Let $A = (a_1 \ \cdots \ a_n)$ be a square matrix and E be an elementary matrix of size n. Then

$$\det EA = \det E \det A.$$

That is

1. If a multiple of one row of A is added to another row to produce a matrix B. Then $\det B = \det A$ (i.e.(也就是) $\det E = 1$ in this case).
2. If two rows of A are interchanged to produce matrix B, then $\det B = -\det A$ (i.e. $\det E = -1$).
3. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$ (i.e. $\det E = k$).

Proof. **Part 1**: We consider 3 types of elementary matrix separately. We first prove:

1. Interchange:$E = E_n(i,j)$(交换第i和第j行). AE is obtained by exchanging the i–th and j–th columns of A. So
$$\det EA = -\det A.$$
2. Scaling: $E = E_n(i;\lambda)$(第i行乘以k). AE is obtained by multiplying the i–th column of A by λ. So
$$\det EA = k \det A.$$
3. Replacement: $E = E_n(i,j;k)$(第j行乘以k后加到第i行上去). So
$$\det EA = \det A.$$

STEP 1: Let us begin with a $2 \times 2 (n = 2)$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So

1. For row replacement, adding row one multiplied by k to row 2, we have $A' = \begin{pmatrix} a & b \\ ra + c & rb + d \end{pmatrix}$. Then we have $E = A'$ in this case and $\det A' = rab + ad - rab - bc = ad - bc = \det A$. The same result can be obtained by adding a multiple of row two to row one.

2. For interchanging two rows in A, A becomes $A' = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$. Hence, $|A| = ad - bc = -(bc - ad) = -|A'|$.

3. For scaling, it is easy to show that the determinant of $A' = \begin{pmatrix} ra & rb \\ c & d \end{pmatrix}$ is $\det A' = k(ad - bc) = k \det A$. The same results can be obtained by scaling the other row.

STEP 2: We now use induction method to prove the rest. Suppose that the theorem is true for determinants of $k \times k$ matrix with $k \geqslant 2$. Now let A be a $k \times k$ identity matrix. Note that the action is only on two rows or only one row. So we can expand $\det EA$ across a row that is unchanged by the action of E, say, row i. Then we have

$$\det EA = \sum_{j=1}^{n} (-1)^{i+j} [EA]_{ij} \det \overline{[EA]}_{ij} = \sum_{j=1}^{n} (-1)^{i+j} [A]_{ij} \det \overline{[EA]}_{ij} \tag{5}$$

Note that the cofactor matrix $\overline{[EA]}_{ij}$ is obtained by performing the same elementary row operation on the cofactor matrix $\overline{[A]}_{ij}$. Hence, we should have: $\det \overline{[EA]}_{ij} = \alpha \det \overline{[A]}_{ij}$, where $\alpha = -1, k, 1$ for interchange, scaling, replacement respectively.

Part 2: We then easily prove that:

1. $\det E = 1$, if E is a row replacement matrix by adding a multiple of one row to another row on an identity matrix I.

2. $\det E = -1$, if E is an interchange matrix by interchanging two rows of identity matrix I.

3. $\det E = k$, if E is a scale matrix by multiplying a row of identity matrix I by a nonzero scalar k.

This is the simple generalization of the proof in Part 1 by setting $A = I_n$.

Part 3: Hence, we finally have $\det EA = \det E \det A$. $\square$

As part of the proof in the above theorem, we have:

THEOREM 8.

1. $\det E = 1$, *if E is a row replacement matrix by adding a multiple of one row to another row on an identity matrix I.*
2. $\det E = -1$, *if E is an interchange matrix by interchanging two rows of identity matrix I.*
3. $\det E = k$, *if E is a scale matrix by multiplying a row of identity matrix I by a nonzero scalar k.*

Examples: Page 193 (见板书)

3.3 Determinants of the Product of Two General Matrices

We first need to prove the following theorem:

THEOREM 9. *The square matrix $A \in \mathbb{R}_{n\times n}$ is invertible iff* $\det A \neq 0$.

Proof.

1. **STEP 1**: If A is invertible, A is equivalent to the identity matrix. That is, there is a series of elementary matrices $E_l, E_{l-1}, \cdots, E_1$ such that $E_l E_{l-1} \cdots E_1 A = I$. So that $\det E_l \det E_{l-1} \cdots \det E_1 \det A = 1$. As $\det E_i \neq 0$, hence $\det A \neq 0$.

2. **STEP 2**: If $\det A \neq 0$, we now prove A is invertible. Assume that A can be reduced to a reduced echelon matrix U. That is, there is a series of elementary matrices $E_l, E_{l-1}, \cdots, E_1$ such that $E_l E_{l-1} \cdots E_1 A = U$. So that $\det E_l \det E_{l-1} \cdots \det E_1 \det A = \det U$. As $\det E_i \neq 0$ and $\det A \neq 0$, so $\det U \neq 0$. Hence each column of U must be a pivot column. As A is a square matrix, so U must be an identity matrix. That is $A \sim I$, and therefore A is invertible. (注意：这里如果U存在一列是非主元列，这意味着那一列是全0，那么可以直接在那一列展开计算U的行列式，从而可以得到U的行列为0的矛盾结果)

□

Now, we reach the main theorem in this subsection.

THEOREM 10.
Let A and B are $n \times n$ matrices. Then $\det AB = \det A \det B$.

Proof. If A is not invertible then neither AB nor A^T is invertible. So $\det AB = \det A \det B = 0$ and $\det A^T = \det A = 0$.

Otherwise, $A \sim I$, so that $A = E_l \cdots E_2 E_1 I$ where $E_1, \ldots, E_l$ are elementary matrices. Then we have

$$
\begin{aligned}
\det AB &= E_l \cdots E_2 E_1 \det B \\
&= E_l \cdots E_2 \det E_1 \det B \\
&\cdots \\
&= \det E_l \cdots \det E_2 \det E_1 \det B \\
&\cdots \\
&= \det E_l \cdots E_2 \det E_1 \det B \\
&= \det E_l \cdots E_2 E_1 \det B \\
&= \det A \det B.
\end{aligned}
$$

□

3.4 More Properties

THEOREM 11. $\det A^{-1} = \frac{1}{\det A}$ *in the case that A is invertible.*

Proof. We have $1 = \det I = \det AA^{-1} = \det A \det A^{-1}$. So $\det A \neq 0$ and $\det A^{-1} = \frac{1}{\det A}$.

□

The following theorem reveal the connection between a determinant function and a linear transformation.

THEOREM 12. *Suppose that the j^{th} column of A is allowed to vary and write*

$$A = [\vec{a}_1, \cdots, \vec{a}_{j-1}, \vec{x}, \vec{a}_{j+1}, \cdots, \vec{a}_n].$$

Define a transformation T from $\mathbb{R}^n$ to $\mathbb{R}$:

$$T(\vec{x}) = \det[\vec{a}_1, \cdots, \vec{a}_{j-1}, \vec{x}, \vec{a}_{j+1}, \cdots, \vec{a}_n].$$

Then we have

$$\begin{aligned} T(c\vec{x}) &= cT(\vec{x}) \\ T(\vec{u}+\vec{v}) &= T(\vec{u}) + T(\vec{v}), \text{ for all } \vec{u}, \vec{v} in \mathbb{R}^n. \end{aligned}$$

Proof. Sketch proof: Compute the determinant from a cofactor expansion of $\det A$ down the j^{th} column. □

4 Computation of Determinants

The final issue we concern with is how to compute the determinant of a given square matrix A. In the following, we give ideas on this computational issue.

1. The determinants of some matrices can be computed readily using properties given in DEFINITION 1 and THEOREM 3&4. Note that these properties and results also held for rows.

   Examples: Textbook P193.

2. We can use any cofactor expansion of a matrix directly to compute the determinant of that matrix, especially, when most of entries of that matrix are zeros.

   Examples: Textbook P.188, P.189.

   However, this method is not efficient in general. It is easy to show that the number of terms of the complete expansion of a determinant of an $n \times n$ matrix is equal to $n!$, which makes the computation impractical when n is large (e.g. $n = 100$).

3. Recall that if A is an $n \times n$ matrix and invertible then $A = E_l \cdots E_1$ where $E_1, \ldots, E_l$ are elementary matrices, otherwise $\det A = 0$. Factorizing an invertible matrix A into $A = E_l \cdots E_1$ can be achieved by transforming A into Reduced Echelon Form (REF) which is I_n. Once we find $E_1, \ldots, E_l$, we have $\det A = \det E_l \cdots \det E_1$.

Young Hare, *by Dürer*