Lecture Note on Linear Algebra
17. Standard Inner Product

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1 What Do You Learn from This Note

Up to this point, the theory we have established can be applied to any vector space in general. However, vector spaces over $\mathbb{R}$ have geometric structure on their own. You probably have already known concepts such as lengths, angles and distances of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Now we shall see how these geometric concepts can be generalized to vector spaces over $\mathbb{R}$, particularly $\mathbb{R}^n$.

Basic Concept: inner product (内积), length (长度), unit vector (单位向量), distance (距离), orthogonality (正交), orthogonal complements (正交补)

2 Inner Product & Length

2.1 Basic Concept

Definition 1 (INNER PRODUCT OR DOT PRODUCT, 内积)**.** Let $\vec{u}$, $\vec{v} \in \mathbb{R}^n$. The standard inner product or dot product $\vec{u} \cdot \vec{v}$ of $\vec{u}$ and $\vec{v}$ is defined to be the real number $\vec{u}^T v$.

Definition 2 (NORM OR LENGTH, 模或长度)**.** Let $\vec{v} \in \mathbb{R}^n$. The norm or length $\|\vec{v}\|$ of $\vec{v}$ is defined to be

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}},$$

which is a non–negative real number.

Definition 3 (UNIT VECTOR, 单位向量)**.** *is a vector whose norm is* 1*. If we are given any non–zero vector* $\vec{v}$*, then* $\frac{\vec{v}}{\|\vec{v}\|}$ *is a unit vector, which is called the unit vector in the same direction as* $\vec{v}$*. The process of creating* $\frac{\vec{v}}{\|\vec{v}\|}$ *is called normalisation.*

Definition 4 (ANGLE)**.** Let $\vec{u}$, $\vec{v} \in \mathbb{R}^n - \{\vec{0}\}$. The angle θ between $\vec{u}$ and $\vec{v}$ is defined to be

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|},$$

which is a number in the interval $[\,0, \pi]$.

Definition 5 (DISTANCE,距离)**.** Let $\vec{u}$, $\vec{v} \in \mathbb{R}^n\}$. The distance $d(\vec{u}, \vec{v})$ between $\vec{u}$ and $\vec{v}$ is defined to be

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

Definition 6 (ORTHOGONALITY,正交)**.** *Two vectors* $\vec{u}$ *and* $\vec{v}$ *in* $\mathbb{R}^n$ *are orthogonal if* $\vec{u} \cdot \vec{v} = 0$*.*

2.2 Some Properties

Theorem 7. Let $\vec{u}$, $\vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then
1. $\vec{u} \cdot \vec{u} \geqslant 0$ and $\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = \vec{0}$;
2. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$;
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$;
4. $\vec{u} \cdot c\vec{v} = c(\vec{u} \cdot \vec{v})$.
(So we say $\mathbb{R}^n$ equipped with dot product is an inner product space.)

Proof. Straightforward. □

Remark: 1) 3. and 4. of THEOREM 2 indicate that dot product is linear on the second operand, and 2. ensures that dot product is also linear on the first operand.

2) We have $\|\vec{v}\| = 0$ iff $\vec{v} = \vec{0}$.
3) For any scalar c, we have $\|c\vec{v}\| = \sqrt{c\vec{v} \cdot c\vec{v}} = \sqrt{c^2}\sqrt{\vec{v} \cdot \vec{v}} = |c|\|\vec{v}\|$.

Example: Textbook P.377.

Theorem 8 (THE CAUCHY–SCHWARZ INEQUALITY)**.** Let $\vec{u}$, $\vec{v} \in \mathbb{R}^n$. Then

$$(\vec{u} \cdot \vec{v})^2 \leqslant (\vec{u} \cdot u)(\vec{v} \cdot \vec{v}), \quad \text{or equivalently,} \quad |\vec{u} \cdot \vec{v}| \leqslant \|\vec{u}\|\|\vec{v}\|.$$

The equality holds if and only if $\vec{u}$ and $\vec{v}$ are linearly dependent.

Proof. **Step 1**. Suppose first that $\vec{u}$ and $\vec{v}$ are linearly independent. Then neither $\vec{u}$ nor $\vec{v}$ is $\vec{0}$. Let t be any scalar. Then $\vec{u} - tv \neq \vec{0}$, so we have

$$\begin{aligned} 0 &< (\vec{u} - tv) \cdot (\vec{u} - tv) \quad [\text{BY 1. OF THEOREM 2.}] \\ &= (\vec{u} \cdot \vec{u}) - t(\vec{u} \cdot \vec{v}) - t(\vec{v} \cdot \vec{u}) + t^2(\vec{v} \cdot \vec{v}) \quad [\text{BY LINEARITY.}] \\ &= (\vec{u} \cdot \vec{u}) - 2t(\vec{u} \cdot \vec{v}) + t^2(\vec{v} \cdot \vec{v}) \quad [\text{BY SYMMETRY.}] \end{aligned}$$

By the arbitraryness of t, we may take $t = \frac{(\vec{u} \cdot \vec{v})}{(\vec{v} \cdot \vec{v})}$ and substitute it into the above inequality to obtain

$$\begin{aligned} 0 &< (\vec{u} \cdot \vec{u}) - 2\frac{(\vec{u} \cdot \vec{v})}{(\vec{v} \cdot \vec{v})}(\vec{u} \cdot \vec{v}) + \frac{(\vec{u} \cdot \vec{v})^2}{(\vec{v} \cdot \vec{v})^2}(\vec{v} \cdot \vec{v}) \\ &= (\vec{u} \cdot \vec{u}) - 2\frac{(\vec{u} \cdot \vec{v})^2}{(\vec{v} \cdot \vec{v})} + \frac{(\vec{u} \cdot \vec{v})^2}{(\vec{v} \cdot \vec{v})} \\ &= (\vec{u} \cdot \vec{u}) - \frac{(\vec{u} \cdot \vec{v})^2}{(\vec{v} \cdot \vec{v})}. \end{aligned}$$

Multiply $(\vec{v} \cdot \vec{v})$ on both side, we get

$$0 < (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2.$$

So $(\vec{u} \cdot \vec{v})^2 < (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v})$.

Step 2. If $\vec{u}$ and $\vec{v}$ are linearly dependent then either $\vec{u} = \lambda\vec{v}$ or $\vec{v} = \lambda\vec{u}$. For both cases, it is obvious that $(\vec{u} \cdot \vec{v})^2 = (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v})$. □

By the Cauchy–Schwarz inequality, for any $\vec{u}$, $\vec{v} \in \mathbb{R}^n - \{\vec{0}\}$ we have

$$-1 \leqslant \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \leqslant 1,$$

Remarks:
1. By definition, we have $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$.
2. By Cauchy–Schwarz inequality, $\theta = 0$ or π iff $\vec{u}$ and $\vec{v}$ are linearly dependent. It is not hard to see that $\theta = 0$ if $\vec{u}$ and $\vec{v}$ are in the same direction and $\theta = \pi$ if $\vec{u}$ and $\vec{v}$ are in the opposite direction.

Theorem 9. Let $\vec{u}$, $\vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then
1. $\|\vec{u}\| \geqslant 0$ and $\|\vec{u}\| = 0$ iff $\vec{u} = \vec{0}$;
2. $\|c\vec{u}\| = |c|\|\vec{u}\|$;
3. $\|\vec{u}+\vec{v}\| \leqslant \|\vec{u}\| + \|\vec{v}\|$ and the equality holds iff $\vec{u}$ and $\vec{v}$ are in the same direction.
(So we say $\mathbb{R}^n$ equipped with $\|\cdot\|$ is a normed space.)

Proof. 1. and 2. have been shown already. For 3. we have

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= \vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v}+2(\vec{u}\cdot\vec{v})\\ &\leqslant \|\vec{u}\|^2+\|\vec{v}\|^2+2\|\vec{u}\|\|\vec{v}\| \quad [\text{BY CAUCHY–SCHWARZ INEQUALITY.}]\\ &= (\|\vec{u}\|+\|\vec{v}\|)^2\end{aligned}$$

So $\|\vec{u}+\vec{v}\| \leqslant \|\vec{u}\|+\|\vec{v}\|$ and the equality holds iff $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|$ iff $\vec{u}$ and $\vec{v}$ are in the same direction. □

Example: Textbook P.378.

Theorem 10. Let $\vec{u}$, $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$. Then
1. $d(\vec{u},\vec{v}) \geqslant 0$ and $d(\vec{u},\vec{v}) = 0$ iff $\vec{u} = \vec{v}$;
2. $d(\vec{u},\vec{v}) = d(\vec{v},\vec{u})$;
3. TRIANGLE INEQUALITY: $d(\vec{u},\vec{w}) \leqslant d(\vec{u},\vec{v}) + d(\vec{v},\vec{w})$.
(So we say $\mathbb{R}^n$ equipped with $d(\,\cdot\,,\cdot\,)$ is a metric space.)

Proof. 1. and 2. are obvious. For 3., we have

$$d(\vec{u},\vec{w}) = \|\vec{u}-\vec{w}\| = \|(\vec{u}-\vec{v})+(\vec{v}-\vec{w})\| \leqslant \|\vec{u}-\vec{v}\|+\|\vec{v}-\vec{w}\| = d(\vec{u},\vec{v})+d(\vec{v},\vec{w}).$$

□

3 Orthogonality: More

Definition 11 (orthogonality,正交)**.** *Vectors $\vec{u}$ and $\vec{v}$ are said to be* ***orthogonal****, written as $\vec{u}\perp\vec{v}$, iff $\vec{u}\cdot\vec{v} = 0$. If neither $\vec{u}$ nor $\vec{v}$ is $\vec{0}$, then $\vec{u}\perp\vec{v}$ is equivalent to $\theta = \frac{\pi}{2}$.*

Theorem 12. Let $\vec{u}$, $\vec{v} \in \mathbb{R}^n$.
1. THE COSINE THEOREM: If neither $\vec{u}$ nor $\vec{v}$ is $\vec{0}$, then

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{u}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta.$$

2. THE PYTHAGOREAN THEOREM: If $\vec{u} \perp \vec{v}$, then

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Proof. 1.

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2(\vec{u} \cdot \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta.$$

2.

$$\|\vec{u} + \vec{v}\|^2 = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + 2(\vec{u} \cdot \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

□

Definition 13 (orthogonal complements, 正交补)**.** *If a vector $\vec{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\vec{z}$ is said to be orthogonal to W. The set of all vector $\vec{z}$ that are orthogonal to W is called the* ***orthogonal complement*** *of W and is denoted by $W^\perp$.*

Theorem 14. *Let W be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is a subspace.*

Proof. Exercise. □

Theorem 15. *Let A be an $m \times n$ matrix. Then $(RowA)^\perp = NulA$ and $(ColA)^\perp = NulA^T$*

THE FLIGHT INTO EGYPT, *by Giotto di Bondone*