Linear Algebra 19
Inner Product Spaces

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Definition 1 (real inner product space)**.** Let V be a vector space over $\mathbb{R}$. An inner product defined on V is a function

$$\langle \cdot\, ,\, \cdot \rangle : V \times V \to \mathbb{R},$$

(that is a function which associates each pair of vectors in V to a scalar) satisfying the following axioms:
for u, v and $w \in V$ and $c \in \mathbb{R}$, we have
1. $\langle u, u\rangle \geqslant 0$ and $\langle u, u\rangle = 0$ iff $u = \vec{0}$;
2. $\langle u, v\rangle = \langle v, u\rangle$;
3. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$;
4. $\langle u, cv\rangle = c\langle u, v\rangle$.
A vector space equipped with an inner product defined as above is called a (real) inner product space.

Example: The space $\mathbb{R}^n$ together with the dot product forms an inner product space which is the standard inner product space.

Example: Let $A \in \mathrm{M}_{m\times n}(\mathbb{R})$ and $\mathrm{rank}(A) = n$. Then the homomorphism $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_A(x) = Ax$ is injective. For $u, v \in \mathbb{R}^n$, define

$$\langle u, v\rangle_A = Au \cdot Av = u^T A^T Av.$$

Then it can be verified easily that $\mathbb{R}^n$ together with $\langle \cdot\, ,\, \cdot \rangle_A$ forms an inner product space.

Example: In general, let V be a vector space over $\mathbb{R}$ and T an injective homomorphism from V to $\mathbb{R}^m$. For $u, v \in V$, define

$$\langle u, v\rangle_T = T(u) \cdot T(v).$$

Then V together with $\langle \cdot , \cdot \rangle_T$ forms an inner product space. In particular, let $x_0, \ldots, x_n$ be distinct real numbers and define

$$T : \mathbb{R}_n[x] \to \mathbb{R}^{n+1}, \quad T(p(x)) = (p(x_0) \ \cdots \ p(x_n))^T,$$

then T is an isomorphism. So

$$\langle p(x), q(x) \rangle_T = T(p(x)) \cdot T(q(x)) = p(x_0)q(x_0) + \cdots + p(x_n)q(x_n)$$

is an inner product defined on $\mathbb{R}_n[x]$ (see Textbook P.429.).

Example: The concept of inner product is not restricted to finite dimensional vector spaces. Let $f(x),\ g(x) \in \mathcal{C}([a, b])$, where $[a, b]$ is a closed interval in $\mathbb{R}$. Define

$$\langle f(x), g(x) \rangle = \int_a^b f(x)g(x)\, dx.$$

Then it is easy to verify that $\langle \cdot , \cdot \rangle$ satisfies the axioms of inner product, hence $\mathcal{C}([a, b])$ together with $\langle \cdot , \cdot \rangle$ is indeed an inner product space.

The theory we developed for the standard inner product space can be translated into the general theory for inner product space by replacing $\mathbb{R}^n$ by a general vector space V over $\mathbb{R}$ and the dot product by an inner product $\langle \cdot , \cdot \rangle$ defined on V. For instance, we have the Cauchy–Schwarz inequality

$$\langle u, v \rangle^2 \leqslant \langle u, u \rangle \langle v, v \rangle$$

holds in general and hence we can define angle between vectors. The concepts of norm, distance, orthogonality, orthonormality, and orthogonal projection etc. can be generalised in the same way without any difficulties. Also, the Gram–Schmidt process and the least–squares problem can also be derived for general inner product spaces and we shall omit the detail discussion for all of them, which are almost identical to those for the standard inner product space.

Autumn Landscape, *by Van Gogh*