Lecture Note on Linear Algebra

8. Matrix Operations

Wei-Shi Zheng,
wszheng@ieee.org，2011

October 18, 2011

1 What Do You Learn from This Note

In the previous lectures, we have seen that matrices play an important role in solving system of linear equations and in studying linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Let $\mathrm{R}^{m\times n}$ denote the set of all $m \times n$ matrices over $\mathbb{R}$. A column vector of dimension n is defined to be an $n \times 1$ matrix. It turns out that $\mathbb{R}^n = \mathrm{R}^{n\times 1}$. Recall that we have defined addition and scalar multiplication on $\mathbb{R}^n$ for any $n \geqslant 1$. We also defined the product of a matrix and a vector. In this note, these operations are extended to the operations between matrices.

Basic concept： diagonal entries(对角线元素), main diagonal(主对角线), zero matrix(零矩阵), square matrix(方阵), diagonal matrix(对角线矩阵), identity matrix(单位矩阵), transpose(转置), matrix inverse(矩阵的逆), elementary matrix (初等矩阵)

Question: Why do we need matrix computation?

1. A linear transformation is corresponding to a unique matrix
2. For two linear transformations T_A and T_B, what is the combination of two linear transform, i.e. $(T_A + T_B)(\vec{x})$?
3. What is compound transformation $T_A \circ T_B(\vec{x})$?

Examples: $T_A(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $T_B(\vec{x}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so what is $T_A \circ T_B(\vec{x})$? (见板书)

2 Some Basic Terminologies for Matrix

Let $A \in \mathrm{R}^{m\times n}$. The entry of A in the i–th row and the j–column is called the (i, j)–entry of A, written $[A]_{ij}$ or a_{ij}. If we are only concerned about the columns of A, A can be written as $(\vec{a}_1 \ \cdots \ \vec{a}_n)$ where a_i represents the i–th column of A.

diagonal entries(对角线元素): The diagonal entries of A are entries a_{11}, a_{22}, . . . , which form the **main diagonal(主对角线)** of A.

zero matrix(零矩阵): If the entries of A are all zero, then A is the zero matrix, which is denoted by $0_{m\times n}$ or simply 0.

square matrix(方阵): If $m = n$, i.e. $A \in \mathrm{R}^{n\times n}$, then A is called a square matrix of size n.

diagonal matrix(对角线矩阵): A square matrix A of size n with all entries being zero except those in the main diagonal is called a diagonal matrix, which can be written as $A = \mathrm{diag}(a_{11}, \ldots, a_{nn})$.

identity matrix(单位矩阵): If $A = \mathrm{diag}(1, \ldots, 1)$, then A is the identity matrix of size n, which is denoted by I_n or simply I. We also write $e_i \in \mathbb{R}^n$ for the i–th column of I_n, so $I_n = (\vec{e}_1 \ \cdots \ \vec{e}_n)$.

3 Some Operations

Let A, $B \in \mathrm{R}^{m\times n}$ and $r \in \mathbb{R}$. The equality, addition and scalar multiplication are defined as follows:

1. **Equality**: We say A and B are equal, written $A = B$, iff $[A]_{ij} = [B]_{ij}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

2. **Addition**: The $m \times n$ matrix $A + B$ such that $[A + B]_{ij} = [A]_{ij} + [B]_{ij}$ is called the sum of A and B.

3. **Scalar Multiplication(数乘)**: The $m \times n$ matrix rA such that $[rA]_{ij} = r[A]_{ij}$ is called the scalar multiple of A by scalar r.

4. **Multiplication**: Let $A \in \mathrm{R}^{m\times n}$ and $B = (\vec{b}_1 \ \cdots \ \vec{b}_p) \in \mathrm{R}^{n\times p}$. The product of A and B, written AB, is defined to be the matrix $(A\vec{b}_1 \ \cdots \ A\vec{b}_p) \in \mathrm{R}^{m\times p}$.

Remark 1:

- AB is defined iff the number of columns of A equals the number of rows of B (矩阵A的列数要与矩阵B的行数相同，这样才能使得AB有定义。).
- **Row-Column Rule(行列法则)**: We can calculate $[AB]_{ij}$ as follows:

$$[AB]_{ij} = \text{the } i\text{–th entry of } A\vec{b}_j = \sum_{k=1}^{n} a_{ik}b_{kj} = (a_{i1} \ \cdots \ a_{in}) \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

Remark 2:

- In general, $AB \neq BA$(矩阵之间的乘法，不像以往数字之间乘法那样，可以互换位置). For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$, where as $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 8 \end{pmatrix}$.
- It is not like the case of real numbers, $AB = 0$ does not imply $A = 0$ or $B = 0$. For instance, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- The cancelation law (消去率) generally fails, that is, $AB = AC$ (or $BA = CA$) does not imply $B = C$. This fact is a consequence of Remarks 2. since $AB = AC$ can be rewritten as $A(B - C) = 0$, by which $B - C = 0$ can not be derived. For instance, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but we do not have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- For square matrix A and a non–negative integer n, we can define the n–th power of A (矩阵A的n次幂) to be

$$A^n = \underbrace{A \quad \cdots \quad A}_{n \text{ times}}.$$

  if $n \geqslant 1$, otherwise $A^0 = I$.

In addition, let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Then the map $T \circ S : \mathbb{R}^p \to \mathbb{R}^m$, $T \circ S(\vec{x}) = T(S(\vec{x}))$ is called the composition of T and S (变换T和S的复合). It is only a routine work to verify that $T \circ S$ is indeed a linear transformation. Now let A and B be the standard matrices for T and S respectively. Then we have $T \circ S(\vec{x}) = T(S(\vec{x})) = T(B\vec{x}) = A(B\vec{x}) = (AB)\vec{x}$. This indicates that AB is exactly the standard matrix for $T \circ S$.

5. **Transpose(转置)**: Let $A \in \mathrm{R}^{m\times n}$. Then the transpose A^T of A is an $n \times m$ matrix such that $[A^T]_{ij} = [A]_{ji}$.

Suppose that A, $B \in \mathrm{R}^{n\times n}$. By definition, AB is defined and $AB \in \mathrm{R}^{n\times n}$, which indicates that $\mathrm{R}^{n\times n}$ is closed under multiplication. The closure of multiplication leads to the following useful operation defined on square matrices.

6. **Matrix inverse(矩阵的逆)**: Let $A \in \mathrm{R}^{n\times n}$. If there is another $B \in \mathrm{R}^{n\times n}$ such that $AB = BA = I_n$, then A is said to be **invertible** (可逆的) and B is called the inverse of A.

Remark 3:

- Not all square matrices are invertible (不是所有矩阵都有逆). A notable case is the zero matrix since for any $A \in \mathrm{R}^{n\times n}$, $0_{n\times n}A = 0_{n\times n} \neq I_n$.
- The inverse of any invertible matrix is unique(矩阵的逆是唯一的). To see this, suppose that matrices B and C are both the inverses of A. Then $B = IB = (CA)B = C(AB) = CI = C$. Normally, the unique inverse of A is denoted by A^{-1}.

4 Some Theorems

4.1 Theorem about Addition Operation

THEOREM 1. Let A, B, $C \in \mathrm{R}^{m\times n}$ and r, $s \in \mathbb{R}$. Then
1. $(A+B)+C = A+(B+C)$ (Additive Associativity，加法结合律);
2. $0+A = A+0$ (Additive Identity);
3. $(-A)+A = A+(-A) = 0$ (Additive Inverseness);
4. $A+B = B+A$ (Additive commutativity, 加法交换律);
5. $r(A+B) = rA+rB$;
6. $(r+s)A = rA+sA$;
7. $r(sA) = (rs)A$;
8. $1A = A$.

4.2 Theorem about Multiplication Operation

THEOREM 2. Let $A \in \mathrm{R}^{m\times n}$. Then
1. $(AB)C = A(BC)$ where $B \in \mathrm{R}^{n\times p}$, $C \in \mathrm{R}^{p\times q}$ (Multiplicative Associativity);
2. $I_m A = AI_n = A$ (Multiplicative Identity);
3. $A(B+C) = AB+AC$ where $B,\ C \in \mathrm{R}^{n\times p}$ (Left Distributivity，左边分配律);
4. $(B+C)A = BA+CA$ where $B,\ C \in \mathrm{R}^{l\times m}$ (Right Distributivity，右边分配律);
5. $r(AB) = (rA)B = A(rB)$ where $r \in \mathrm{R}$, $B \in \mathrm{R}^{n\times p}$.

Proof. 1. We have

$$[(AB)C]_{ij} = \sum_{v=1}^{p}[AB]_{iv}c_{vj} = \sum_{v=1}^{p}\Big(\sum_{u=1}^{n}a_{iu}b_{uv}\Big)c_{vj} = \sum_{v=1}^{p}\sum_{u=1}^{n}a_{iu}b_{uv}c_{vj}$$

and

$$[A(BC)]_{ij} = \sum_{u=1}^{n}a_{iu}[BC]_{uj} = \sum_{u=1}^{n}a_{iu}\Big(\sum_{v=1}^{p}b_{uv}c_{vj}\Big) = \sum_{u=1}^{n}\sum_{v=1}^{p}a_{iu}b_{uv}c_{vj}.$$

So $[(AB)C]_{ij} = [A(BC)]_{ij}$ and the result follows.
2.–5. are easy and left as exercises. □

The next theorem tells us, we can describe the row operation on a matrix A by multiplication of a matrix E which is called the elementary matrix and A.

Theorem 3. *Given a $m \times n$ matrix A, we have*

1. ***Scaling**: Given a $m \times m$ matrix E, for some i_0,*

$$[E]_{ij} = \begin{cases} 1, & i = j \text{ and } i \neq i_0 \text{ and } j \neq j_0; \\ k, & i = i_0 \text{ and } j = i_0; \\ 0, & \text{otherwise.} \end{cases}$$

then EA will multiply a constant value k on the row i_0 of matrix A.

e.g. $E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. ***Interchange**: Given a $m \times m$ matrix E, for some $i_0, j_0, i_0 \neq j_0$,*

$$[E]_{ij} = \begin{cases} 1, & i = j \text{ and } i \neq i_0 \text{ and } j \neq j_0; \\ 1, & i = i_0 \text{ and } j = j_0; \\ 1, & i = j_0 \text{ and } j = i_0; \\ 0, & \text{otherwise.} \end{cases}$$

then EA will interchange row i_0 and row j_0 in matrix A.

e.g. $E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3. ***Replacement**: Given a $m \times m$ matrix E, for some $i_0, j_0, i_0 \neq j_0$,*

$$[E]_{ij} = \begin{cases} 1, & i = j \text{ and } i \neq i_0 \text{ and } j \neq j_0; \\ k, & i = i_0 \text{ and } j = j_0; \\ 0, & \text{otherwise.} \end{cases}$$

, then EA will replace row i_0 of matrix A by the addition of the original row i_0 and a multiple of the row j_0 of matrix A.

e.g. $E = \begin{pmatrix} 1 & k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Example: 见板书.

Formally, we call all matrices E in Theorem 3 as **elementary matrix** (初等矩阵). This theorem tells us each row operation is corresponding to an elementary matrix. Row operation can be realized by matrix multiplication.

注：上面的定理正好告诉我们对一个矩阵A三种基本行变换可以通过相应的初等矩阵E左乘以矩阵A实现。

Moreover, the elementary is actually invertible according to the last theorem as well. Let us together go through the following example:
Example:

$$E_3(2,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad E_3(3;5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}; \quad E_3(1,3;8) = \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the inverse is: Example:

$$E_3(2,3)^{-1} = E_3(2,3); \quad E_3(3;5)^{-1} = E_3(3;5^{-1}); \quad E_3(1,3;8)^{-1} = E_3(1,3;-8).$$

4.3 Theorem about Transpose Operation

Theorem 4. Let $A \in \mathrm{R}^{m\times n}$. Then
1. $(A^T)^T = A$;
2. $(A+B)^T = A^T + B^T$ where $B \in \mathrm{R}^{m\times m}$;
3. $(rA)^T = rA^T$ where $r \in \mathbb{R}$;
4. $(AB)^T = B^T A^T$ where $B \in \mathrm{R}^{n\times p}$.

Proof. 1.–3. are straightforward. For 4., we have

$$[(AB)^T]_{ij} = [AB]_{ji} = \sum_{k=1}^{n} [A]_{jk}[B]_{ki} = \sum_{k=1}^{n} [B^T]_{ik}[A^T]_{kj} = [B^T A^T]_{ij},$$

and the result follows. □

4.4 Theorem about Inverse Operation

4.4.1 Inverse and Multiplication

THEOREM 5. Let $A, B \in \mathrm{R}^{n\times n}$ be invertible. Then
1. $(A^{-1})^{-1} = A$;
2. $(AB)^{-1} = B^{-1}A^{-1}$;
3. $(A^T)^{-1} = (A^{-1})^T$.

Proof. 1. and 3. are obvious.
2. We have

$$(AB)(B^{-1}A^{-1}) = (A(BB^{-1}))A^{-1} = (AI)A^{-1} = AA^{-1} = I.$$

Similarly, $(B^{-1}A^{-1})(AB) = I$. So AB is invertible and the inverse is $B^{-1}A^{-1}$. □

Remark: By THEOREM 4, 2., if A is invertible then for any integer $n \geqslant 0$, A^n is invertible. We normal write A^{-n} for $(A^n)^{-1} = (A^{-1})^n$.

4.4.2 Solution of Matrix Equation If A is invertible

Question: Given that A is invertible, what can we say about the matrix equation $A\vec{x} = \vec{b}$?

THEOREM 6. Let $A \in \mathrm{R}^{n\times n}$ be invertible. Then for any $\vec{b} \in \mathbb{R}^n$, the matrix equation $A\vec{x} = \vec{b}$ has a unique solution $A^{-1}\vec{b}$.

Proof. Since $A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}$, so $A^{-1}\vec{b}$ is indeed a solution. Suppose that $\vec{x}_0$ is a solution, then $A\vec{x}_0 = \vec{b}$. It follows that $A^{-1}(A\vec{x}_0) = A^{-1}\vec{b}$, which gives $\vec{x}_0 = A^{-1}\vec{b}$. This shows the uniqueness of the solution. □

4.4.3 Connection to Row Reduction Algorithm

We first need to prove the following theorem.

THEOREM 7. If $A \in \mathrm{R}^{n\times n}$ is invertible then $A \sim I$.

Proof.

1. Step 1: We first need to prove if $A \in \mathrm{R}^{n\times n}$ is invertible, then $A \sim I$.
   By THEOREM 6, the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\vec{x}) = A\vec{x}$ is surjective. As a result, $\mathrm{Span}\{\vec{a}_1, \dots, \vec{a_n}\} = \mathbb{R}^n$, which indicates that A has a pivot position in every row (the number of pivots $= n$). But A is a square matrix, it turns out that every column of A is a pivot column, i.e. $A \sim I$.

2. Step 2: We next prove if $A \sim I$, then $A \in \mathrm{R}^{n\times n}$ is invertible. The key idea to construct the inverse of matrix A.

   Since $A \sim I$, then we have a series of matrices:

   $$A = A_0 \xrightarrow{op_1} A_1 \xrightarrow{op_2} \cdots \xrightarrow{op_l} A_l = I.$$

   By the preceding discussion, $A_i = E_i A_{i-1}$ where E_i is the elementary matrix associated to op_i for all i. So

   $$I = A_l = E_l A_{l-1} = \cdots = E_l E_{l-1} \cdots E_1 A.$$

   Since E_i is invertible, we can multiply both sides of the above equation by $(E_l E_{l-1} \cdots E_1)^{-1}$ on left to obtain

   $$(E_l E_{l-1} \cdots E_1)^{-1} I = A.$$

   So

   $$A = (E_l E_{l-1} \cdots E_1)^{-1} = E_1^{-1} \cdots E_{l-1}^{-1} E_l^{-1}.$$

□

In other words, the above theorem can be described in a different way as follows:

THEOREM 8. Let $A \in \mathrm{R}^{n\times n}$. Then following statements are equivalent:
1. A is invertible.
2. $A \sim I$.
3. A is the product of some elementary matrices.

By THEOREM 8 and the preceding discussion, we have

COROLLARY 9. Let A, $B \in \mathrm{R}^{m\times n}$. Then $A \sim B$ iff there is an invertible $T \in \mathrm{R}^{n\times n}$ such that $TA = B$.

In practice, we can compute A^{-1} via elementary row operations. From the proof of THEOREM 7, we have

$$A^{-1} = E_l E_{l-1} \cdots E_1 = E_l E_{l-1} \cdots E_1 I.$$

So

$$I \xrightarrow{op_1} E_1 I \xrightarrow{op_2} \cdots \xrightarrow{op_l} E_l E_{l-1} \cdots E_1 I = A^{-1}.$$

So $op_1, \ldots, op_l$ transform A, I to I, A^{-1} simultaneously, i.e.

$$(A\ I) \xrightarrow{op_1} \cdots \xrightarrow{op_l} (I\ A^{-1}).$$

Example: Textbook P.124.
注：我们最后回头来介绍怎么计算。

4.4.4 Connection to Linear Transformation

THEOREM 10. Let $A = (\vec{a}_1 \ \cdots \ \vec{a}_n) \in \mathrm{R}^{n\times n}$. Then following statements are equivalent:
1. A is invertible.
2. $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\vec{x}) = A\vec{x}$ is surjective.
3. $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\vec{x}) = A\vec{x}$ is injective.

Proof. 1. ⇒ 2. and 1. ⇒ 3. are established in THEOREM 5.
2. ⇒ 1. Again, by THEOREM 4 in lecture note 4, A has a pivot position in every row since $\mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \mathbb{R}^n$. So every column of A is a pivot column. It follows that $A \sim I$ since I is the unique REF of A. So 1. holds.
3. ⇒ 1. By assumption, $A\vec{x} = \vec{0}$ has only one solution, namely $\vec{0}$. It follows that every column of A is a pivot column since no free variable for $A\vec{x} = \vec{0}$. So $(A\ \vec{0}) \sim (I\ \vec{0})$, i.e. $A \sim I$. So 1. holds. □

THEOREM 11. Let $A \in \mathrm{R}^{n\times n}$. Then following statements are equivalent:
1. A is invertible.
2. There exists $B \in \mathrm{R}^{n\times n}$ such that $AB = I$.
3. There exists $B \in \mathrm{R}^{n\times n}$ such that $BA = I$.

Proof. 1. ⇒ 2. and 1. ⇒ 3. are obvious since we can take A^{-1} for B.
2. ⇒ 1. Consider $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\vec{x}) = A\vec{x}$. For any $\vec{b} \in \mathrm{R}^{n\times n}$, we have $T_A(B\vec{b}) = A(B\vec{b}) = I\vec{b} = \vec{b}$. So T_A is surjective and the result follows.
3. ⇒ 1. Consider $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\vec{x}) = A\vec{x}$. Suppose that $T_A(x_1) = T_A(x_2)$, namely $A\vec{x}_1 = A\vec{x}_2$. Then we have $BA\vec{x}_1 = BA\vec{x}_2$, which is followed by $\vec{x}_1 = \vec{x}_2$. So T_A is injective and the result follows. □

A simple consequence of THEOREM 10 is

COROLLARY 12. Let A, $B \in \mathrm{R}^{n\times n}$. Then if $AB = I$ (or $BA = I$) then $B = A^{-1}$.

Finally, we introduce the concept of invertible linear transformations for which the standard matrices are exactly those being invertible.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$. If there is another $S : \mathbb{R}^n \to \mathbb{R}^n$ such that for any $\vec{x} \in \mathbb{R}^n$, $T \circ S(\vec{x}) = S \circ T(\vec{x}) = (\vec{x})$, then T is said to be **invertible** and S, written T^{-1} is called the inverse of T.

THEOREM 13. Linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible iff the standard matrix A for T is invertible. Also, $T^{-1}(\vec{x}) = A^{-1}\vec{x}$ in the case that T is invertible.

Proof. Suppose that T is invertible and $T^{-1}(\vec{x}) = B\vec{x}$. Then we have $T \circ T^{-1}(\vec{x}) = T(T^{-1}(\vec{x})) = A(B\vec{x}) = (AB)x$. On the other hand, $T \circ T^{-1}(\vec{x}) = (\vec{x}) = I(\vec{x})$. So both AB and I are standard matrices of $T \circ T^{-1}$. By uniqueness of standard matrix, we must have $AB = I$. So A is invertible and $B = A^{-1}$.

Conversely, suppose that A is invertible. Define $S(\vec{x}) = A^{-1}\vec{x}$. Then $T \circ S(\vec{x}) = T(S(\vec{x})) = A(A^{-1}\vec{x}) = I\vec{x} = \vec{x}$ and $S \circ T(\vec{x}) = S(T(\vec{x})) = A^{-1}(A\vec{x}) = I\vec{x} = \vec{x}$. So T is invertible with inverse S. □

5 A Method for Computing the Inverse of a Matrix

Strategy: Row reduce the augmented matrix $[A \quad I]$. If A is row equivalent to I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, A does not have an inverse.

Tool: Row reduction algorithm and Theorem 7.

Example: Find the inverse of the matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$ if it exists.

Solution:

$$[A\ I] = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 1 & 1 \end{pmatrix}$$

$\rightarrow$ Row Reduction Algorithm $\rightarrow \begin{pmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4-1 & \\ 0 & 1 & 1 & 3/2 & -2 & 1/2 \end{pmatrix}$. Hence,

$A^{-1/} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4-1 & \\ 3/2 & -2 & 1/2 \end{pmatrix}$. (详见板书)

Reference

David C. Lay. Linear Algebra and Its Applications (3rd edition). Pages 105∼134

Saint George and the Dragon, *by Rubens*