Lecture Note on Linear Algebra
13. Vector Spaces and Subspaces

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1 What Do You Learn from This Note

Example: Define $\vec{x} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}$, $\vec{z} = \begin{pmatrix} 7 \\ 8 \\ -9 \end{pmatrix}$. Please verify:
(1) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$;
(2)$(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$;
(3) $\vec{0} + \vec{x} = \vec{x}$;
(4) $2(\vec{x} + \vec{y}) = 2\vec{x} + 2\vec{y}$;
(5) $(3 + 5)\vec{x} = 3\vec{x} + 5\vec{x}$;
(6) $3 \cdot (5\vec{x}) = (3 \times 5)\vec{x}$;
(7) $1 \cdot \vec{x} = \vec{x}$.

Question: The above an be valid for each two vectors. So what is the formal concept for combination of set of vectors and its operations? Can it be generalized? That is the vector space (向量空间)we are going to present. (也就是说，向量空间定义有两个：(1)向量的集合；(2)定义在这个向量集合上的运算)

注：在我们前面感性认识的基础上，本节将从理论体系上较完整地介绍向量空间的理论。本节会比较理论，包括一个向量空间的定义怎样才合理的问题。

另外，书本上的向量，vector，一词可能不是指$\mathbb{R}^n$空间中的向量，是一个数学的广义定义。请大家听课的要留神以下两点：

1. 本学期课程，我们所学习的向量空间的定义基本上是定义在欧式空间上(i.e. $\mathbb{R}^n$)。

2. 课本上的向量空间V有时可能指的是欧式空间$\mathbb{R}^n$中的一个子集/局部。那么这时候的问题就是这个子集能叫向量空间吗？

Basic Concept: Vector Space(向量空间), Subspace(子空间), null space(零空间), column space(列空间)

2 Vector Space

In general, we can abstract and generalize the example at the beginning to formulate the concept of vector spaces as follows:

注：下面的定义是一个很广泛的定义，这里的向量空间不单是指我们平常所指的$\mathbb{R}^n$空间(或通常称为欧式空间，Euclidean space)。定义在欧式空间中的向量我们叫欧式向量(Euclidean vector)。大家如果要理解下面的定义，不妨把V看成是$\mathbb{R}^n$来理解会很好。

Definition 1 (vector space). A vector space V is referred to a **non–empty set** whose elements are called vectors together with two operations known as **addition** '+' and **scalar multiplication** '$\cdot$' satisfying the following axioms for all vectors $\vec{u}, \vec{v}, \vec{w}$ and scalars c, d:
(1) $(\vec{u} + \vec{v}) \in V$;
(2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Additive Associativity, 加法结合率);
(3) there exists $\vec{0} \in V$, $\quad \vec{0} + \vec{u} = \vec{u} + \vec{0}$ (Additive Identity, 加法单位元);
(4) there exists $-\vec{u} \in V$, $\quad (-\vec{u}) + \vec{u} = \vec{u} + (-\vec{u}) = \vec{0}$ (Additive Inverse, 加法逆元);
(5) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Additive commutativity, 加法交换律);
(6) $c\vec{u} \in V$;
(7) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$;
(8) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$;
(9) $c(d\vec{u}) = (cd)\vec{u}$;
(10) $1\vec{u} = \vec{u}$.

注：上面向量空间的定义可以归纳为2条：

1. 加法运算闭合, (1)-(5);
2. 乘法运算闭合, (6)-(10);

Remarks:

1. The vector $\vec{0}$ occurred in (2) is called the zero vector (零向量) of V, which is uniquely determined in V since if both $\vec{0}$ and $\vec{0}'$ are zero vectors, then $\vec{0} = \vec{0} + \vec{0}' = \vec{0}'$.

2. For each $\vec{u} \in V$, the vector $-\vec{u}$ occurred in (3) is also uniquely determined (easy exercise).

3. We have $\quad c\vec{0} = \vec{0}, \quad 0\vec{u} = \vec{0}, \quad -\vec{u} = (-1)\vec{u}$ (easy exercise).

4. We have $c\vec{u} = \vec{0}$ implies $c = 0$ or $\vec{u} = \vec{0}$. To see this, we suppose that $c \neq 0$, then $\vec{u} = (c^{-1}c)\vec{u} = c^{-1}(c\vec{u}) = c^{-1}\vec{0} = \vec{0}$.

2.1 Examples

Example: For any $n \in \mathbb{Z}^+$, $\mathbb{R}^n$ (or $\mathbb{C}^n$) forms a vector space over $\mathbb{R}$ (or $\mathbb{C}^n$) under vector addition and scalar multiplication.

Example: Consider the set $\mathbb{C}$ of complex numbers. Since $\mathbb{R} \subset \mathbb{C}$, so we may define the scalar product of $c \in \mathbb{R}$ and $z \in \mathbb{C}$ as the usual product cz of complex numbers. Then $\mathbb{C}$ forms a vector space over $\mathbb{R}$ under usual addition and scalar multiplication.

Example: P217, Example 2.

注：事实上，向量空间的例子还很多，还可以扩展到函数的空间上，见课本219，Example 5。以后有一门课叫泛函分析，上面有更多的介绍。由于大家还没有学更深的数学，我建议课本上的218的Example 4-5不要求掌握，但鼓励大家先了解。我们这个学期的课，都基本针对$\mathbb{R}^n$空间。

3 Subspace (子空间)

Let V be any vector space. A particular type of subsets of V called subspaces plays an important role in the study of the theory of vector spaces.

Definition 2 (subspace). Let V be a vector space and H a non–empty subset of V. Then H is called a subspace of V if H is itself a vector space under the same addition and scalar multiplication on V.

Question: How to determine whether a subset is a subspace? The following theorem tells us.

Theorem 3. Let V be a vector space and H a subset of V. Then H is a subspace iff the following conditions hold:
(i) $\vec{0} \in H$;
(ii) for any $\vec{u}, \vec{v} \in H$, $\vec{u} + \vec{v} \in H$; (i.e. 加法闭合)
(iii) for any $\vec{u} \in H$, $c \in \mathbb{R}$, $c\vec{u} \in H$ (i.e. 乘法闭合).

Proof. It is straightforward to verify that Axioms (1)—(10) hold for H. Hence H forms a vector space under addition and scalar multiplication. □

3.1 Examples

Example: Let V be any vector space. Then the zero space $\{\vec{0}\}$ is a subspace of V. Also, V is itself a subspace of V. Both $\{\vec{0}\}$ and V are called the trivial subspaces of V.

Example: If we regard $\mathbb{C}$ as a vector space over $\mathbb{R}$. Then $\mathbb{R}$ is a subspace of $\mathbb{C}$.

Example: Let m, $n \in \mathbb{R}^+$ and $m \leqslant n$. Define

$$H = \{(\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_m\ 0\ \cdots\ 0)^T \in \mathbb{R}^n\}.$$

Then H is a subspace of $\mathbb{R}^n$.

Example: The following is very interesting:

1. The vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$

2. $H = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} : s, t \text{ are real.} \right\}$

注：这最后的一个例子有助于大家理解subspace和vector space的概念。板书解释。

3.2 Span and Subspace

As we did for $\mathbb{R}^n$, we can define linear combination of vectors in any vector space.

注：大家可能会奇怪为什书本P232又再重新定义Linear combination。让我们与书本P77页的定义比较，它们的差别在于P232页的定义是在任何的向量空间V上定义，该向量空间可能不是我们熟知的$\mathbb{R}^n$，或许是$\mathbb{R}^n$的一个子集，更可能是抽象的空间，比如函数空间。在本课程我们只要求大家掌握V是$\mathbb{R}^n$和子集的情形。但我希望大家有那样的概念：向量空间在数学上是一个很广义的概念。当你们以后学到更深入的数学知识后，我希望你们能再次翻开这本书，慢慢再咀嚼。

Definition 4 (linear combination). Let V be any vector space. Given $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Then we can define a new vector

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n v_n \Big(= \sum_{i=1}^{n} c_i v_i \Big),$$

which is called a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with coefficients (or weights) $c_1, c_2, \ldots, c_n$.

The spanning set of a list of vectors can also be defined in a similar way.

Definition 5 (span). Let V be any vector space and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$. Then the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is called the set generated (or spanned) by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and is denoted by $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, which is a subset of V.

Theorem 6. Let V be any vector space and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$. Then $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a subspace of V

Proof. The proof is straightforward. Just directly use Theorem 3. (板书解释) $\square$

4 Null Space & Column Space

Definition 7 (Null space(零空间)). The null space of an $m \times n$ matrix A, denoted by Nul$A$, is the set of all solutions to the matrix equation $A\vec{x} = \vec{0}$. That is:

$$\text{Nul}A = \{\vec{x} : \ \vec{x} \in \mathbb{R}^n, \ A\vec{x} = \vec{0}\}. \tag{1}$$

Definition 8 (Column Space(列空间)). The column space of an $m \times n$ matrix A, denoted by Col$A$, is the set of all linear combinations of the columns of A. That is if $A = [\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n]$, then:

$$\text{Col}A = Span\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}. \tag{2}$$

Question: Are the null space and column space subspaces? Yes, they are. The following theorems validate this claim.

Theorem 9. *The null space of an $m \times n$ matrix A is a subspace of* $\mathbb{R}^n$

Proof. The proof is straightforward. Just directly use Theorem 3 as follows:

1. $\vec{0} \in \text{Nul}A$, as $A\vec{0} = \vec{0}$;

2. If $\vec{u}, \vec{v} \in \text{Nul}A$, then $\vec{u} + \vec{v} \in \text{Nul}A$, as $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}) = \vec{0}$;

3. If c is any scalar, then $c\vec{u} \in \text{Nul}A$, as $A(c\vec{u}) = c(A\vec{u}) = c\vec{0} = \vec{0}$.

□

Theorem 10. *The null space of an $m \times n$ matrix A is a subspace of* $\mathbb{R}^n$

Proof. This is a direct result of Theorem 6. □

Theorem 11. *$ColA = \{\vec{b} : \vec{b} = A\vec{x}$ for some $\vec{x}$ in $\mathbb{R}^n\}$.*

Example: 重点掌握课本p228页Example 5

Question: What are the differences between null space and column space? Please find it on Page 232. Several Points are emphasized here:

1. Given a specific vector $\vec{v}$, it is easy to tell if $\vec{v}$ is in Nul$A$. Just compute $A\vec{v}$;

   In comparison, it may take time to tell if $\vec{v}$ is in Col$A$ by performing Row operations on $[A\vec{v}]$.

2. Nul$A = \{\vec{0}\}$ if and only if the equation $A\vec{x}$ has only the trivial solution.

   In comparison, Col$A = \mathbb{R}^m$ if and only if the equation $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^m$.

5 Kernel and Range of a Linear Transformation

下面是两条比较广义的定义，大家理解的时候可以把V看成是$\mathbb{R}^n$，把W看成是$\mathbb{R}^m$.

Definition 12 (linear transformations). Let V, W be both vector spaces. A transformation $T : V \to W$ is called linear if for any vectors v, v_1, $v_2 \in V$ and scalar c, we have
1. $T(v_1 + v_2) = T(v_1) + T(v_2)$ (Preservation of Addition);
2. $T(cv) = cT(v)$ (Preservation of Scalar Multiplication).

Definition 13 (Kernel of T). Let $T : V \to W$ be a linear transformation. Then the *kernel (null space)* of T, written as Ker(T), is defined to be the set of all $\vec{u}$ in V such that $T(\vec{u}) = \vec{0}$, where $\vec{0}$ here is the zero vector in W. That is:

$$\text{Kernel of } T = \{\vec{x} : \ \vec{x} \in V, \ T(\vec{x}) = \vec{0}\}$$

Definition 14 (Range of T). The range of T is the set of all vectors in W of the form $T(\vec{x})$ for some $\vec{x}$ in V. That is:

$$\text{Range of } T = \{\vec{b} : \ \vec{b} \in W, \ \vec{b} = T(\vec{x}) \text{ for some } \vec{x} \text{ in } V\}$$

注：当V是$\mathbb{R}^n$，W是$\mathbb{R}^m$时，根据我们前面所学，这时候T是与一个矩阵变换A相对应，所以T的kernel space就是A的null space，T的range space就是A的column space.

THE PEASANT WEDDING (农民的婚礼), *by Bruegel the elder*