Lecture Note on Linear Algebra

15. Dimension and Rank

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1 What Do You Learn from This Note

We still observe the unit vectors we have introduced in Chapter 1:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1}$$

We know the above are the basis (specially the standard basis) of $\mathbb{R}^3$. However, we still have to answer the following question:

Question: Why there are three basis in $\mathbb{R}^3$?

Basic Concept： dimension(维数), rank(秩)

2 Dimension

Theorem 1. Let V be a vector space with basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$. Then any subset of V containing more than n vectors is linearly dependent.

Proof. Let $p \in \mathbb{N}$ and $p > n$. Assume that $\{\vec{u}_1, \ldots, \vec{u}_p\} \subseteq V$. As $\mathcal{B}$ is the basis, so we have for each $\vec{u}_i$ there exists a coefficient vector $\vec{a}_i \in \mathbb{R}^n$ such that

$$\vec{u}_i = B\vec{a}_i,\ B = [\vec{b}_1, \ldots, \vec{b}_n].$$

Let $A = [\vec{a}_1, \ldots, \vec{a}_p], U = [\vec{u}_1, \ldots, \vec{u}_p]$. Then

$$U = BA.$$

The if there is a series of weight $c_1, \cdots, c_p$ such that $c_1\vec{u}_1 + \cdots + c_p\vec{u}_p = \vec{0}$, that is

$$U\vec{c} = \vec{0},$$

then we have

$$BA\vec{c} = \vec{0}.$$

As the number of rows is smaller than the number of columns of BA, so that is there are non-pivot columns in BA. This leads to the matrix equation $BA\vec{c}$ has non-trivial solution. So there exists non-zeros $\vec{c}$ such that $U\vec{c} = \vec{0}$. That is $\vec{u}_1, \ldots, \vec{u}_p$ are linearly dependent. □

Theorem 2. Let V be a vector space with bases $\mathcal{B}$, $\mathcal{C}$ of sizes m, $n \in \mathbb{N}$ respectively. Then $m = n$. That is every basis of V has the same size.

Proof. Since $\mathcal{C}$ is linearly independent, by Theorem 1, we must have $m \geqslant n$. Similarly, $n \geqslant m$. So $m = n$. □

Definition 3 (dimension)**.** Let V be a vector space with a finite subset as its basis. Then the size of its basis is called the dimension of V and is denoted by $\dim V$, and we say V is *finite-dimensional*. Otherwise, if V can not be spanned by a finite set then V is *infinite-dimensional*.

Remark:
(1) $\dim \mathbb{R}^n = n$.
(2) The space with all polynomials (多项式空间) is infinite-dimensional.

3 Subspace & Dimension

Theorem 4. Let V be a vector space of dimension $n \in \mathbb{N}$. Then any linearly independent subset $\{\vec{v}_1, \ldots, \vec{v}_m\}$ of V can be extended to a basis of V.

Proof. We prove this theorem by the following steps:

1. When $m = n$, because if $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_m\} = V$ then $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is a basis of V.

2. When $m < n$ $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_m\} \subset V$ and we can pick $\vec{v}_{m+1} \in V - \text{Span}\{\vec{v}_1, \ldots, \vec{v}_m\}$. We claim that $\{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}\}$ is linearly independent. Assume this is not the case, then there exist $c_1, \ldots, c_m, c_{m+1}$ which are not all zero such that $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m + c_{m+1}\vec{v}_{m+1} = 0$. If $c_{m+1} = 0$ then $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is linearly dependent, otherwise $\vec{v}_{m+1} \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_m\}$. Both cases contradict the assumptions we made. So $\{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}\}$ is linearly independent.

3. If $\{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}\}$ spans V then we obtain a basis, otherwise repeat this process again and finally, we must obtain a linearly independent set containing $\{\vec{v}_1, \ldots, \vec{v}_m\}$ spans V, that is a basis.

□

Theorem 5. *Let H be a subspace of a finite-dimension vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and*

$$dimH \leqslant dimV$$

Theorem 6. Let V be a vector space of dimension $p(\geqslant 1)$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Proof. By theorem 4, any linearly independent set S of size $q(\leqslant p)$ can be expanded to the basis of V. This implies, if $q = p$, then S must be the basis of V. □

Theorem 7. *The dimension of $NulA$ is the number of free variables in the equation $A\vec{x} = \vec{0}$, and the dimension of $ColA$ is the number of pivot columns in A.*

4 Rank of a Matrix A

4.1 Row Space

Definition 8 (Row Space)**.** *If A is an $m \times n$ matrix. The set of all linear combinations of the row vectors is called the row space of A, denoted by $RowA$.*
From another point of view, the rows of A are identical to the columns of A^T, so the row space of A can also be written as $ColA^T$.

Theorem 9. *If two $m \times n$ matrices A and B are row equivalent, their row spaces are the same.*

Proof. Since A and B are row equivalent, there exists a invertible matrix G such that

$$B = GA.$$

That is

$$B^T = A^T G^T = A^T F, \quad F = G^T.$$

By using the matrix partition theory, let $A^T = [\text{Col}_1 A^T, \cdots, \text{Col}_m A^T]$, $F = [\text{Col}_1 F, \cdots, \text{Col}_m F]$, then

$$\begin{aligned} & [\text{Col}_1 B^T, \cdots, \text{Col}_i B^T, \cdots, \text{Col}_m A^T] \\ =& B^T \\ =& A^T F \\ =& [A^T \text{Col}_1 F, \cdots A^T \text{Col}_i F, \cdots, A^T \text{Col}_m F]. \end{aligned} \tag{2}$$

That is

$$\begin{aligned} & \text{Col}_i B^T \\ =& A^T \text{Col}_i F \\ =& [\text{Col}_1 A^T, \cdots, \text{Col}_m A^T]([F]_{1i}, \cdots, [F]_{mi})^T \\ =& [F]_{1i} \text{Col}_1 A^T + \cdots + [F]_{mi} \text{Col}_m A^T. \end{aligned} \tag{3}$$

Since $(\text{Row}_i A)^T = \text{Col}_i A^T$ and $(\text{Row}_i B)^T = \text{Col}_i B^T$, so

$$\text{Row}_i B = [F]_{1i} \text{Row}_1 A + \cdots + [F]_{mi} \text{Row}_m A.$$

That is $\text{Row}B \subseteq \text{Row}A$.

Conversely, by changing the roles of A and B in the above, we can also have $\text{Row}B \subseteq \text{Row}A$. So, $\text{Row}B = \text{Row}A$. $\square$

Theorem 10. *If two $m \times n$ matrices A and B are row equivalent, and B is in echelon form, then the nonzero rows of B form a basis for the row space of A as well as for that of B*

4.2 Definition and Properties of Rank

Definition 11 (column & row rank)**.** Let A be a $m \times n$ matrix. The column rank of A is defined to be $\dim \mathrm{Col}(A)$ and the row rank of A to be $\dim \mathrm{Row}(A)$.

Remark: The dimension of the null space is sometimes called the **nullity** of A.

Theorem 12 (Rank Theorem)**.** *The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. Also, it holds that:*

$$rank A + dim\ Nul A = n. \tag{4}$$

Proof. We know
(1) rank$A$ is the number of pivot columns in A. That is if B is an echelon form of A. i.e. rank$A$ is the number of pivot positions in B.
(2) Each pivot position corresponds to a nonzero row, and these rows form a basis for the row space of A, so the rank of A is also the dimension of the row space.
(3) The dimension of Nul$A$ equals the number of free variables in the equation $A\vec{x} = \vec{0}$. That is the dimension of Nul$A$ is the number of columns of A that are not pivot columns.
(4) Number of pivot columns + number of non-pivot columns = number of columns, i.e.

$$\mathrm{rank} A + \dim\ \mathrm{Nul} A = n.$$

□

5 Computation of Null Space and Rang Space

We are now discussing how to compute bases of $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$ given $A \in \mathbb{R}^{m\times n}$.

Computation of Column Space: Let $A = (\vec{a}_1 \ \cdots \ \vec{a}_n) \in \mathbb{R}^{m\times n}$. Then there exists an invertible matrix $B \in \mathbb{R}^{m\times m}$ such that BA is in REF. Let $B\vec{a}_{i_1}, \ldots, B\vec{a}_{i_r}$ be all the pivot columns of BA. Then it is obvious that $\{B\vec{a}_{i_1}, \ldots, B\vec{a}_{i_r}\}$ is a basis of $\mathrm{Col}(BA)$. Since B is invertible and by THEOREM 15.4, $\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_r}\}$ forms a basis of $\mathrm{Col}(A)$, that is the set of pivot columns of A is a basis of $\mathrm{Col}(A)$.

Computation of Null Space: We also know that $\mathrm{Nul}(A)$ is exactly the solution set of the equation $A\vec{x} = \vec{0}$. So by solving $Ax = \vec{0}$ using ROW REDUCTION ALGORITHM, we obtain $\vec{v}_1, \ldots, \vec{v}_{n-r} \in \mathrm{Nul}(A)$, such that $\mathrm{Nul}(A) = \{x_{j_1}\vec{v}_1 + \cdots + x_{j_{n-r}}\vec{v}_{n-r} \,|\, x_{j_1}, \ldots, x_{j_{n-r}} \in \mathbb{R}\} = \mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_{n-r}\}$, where $x_{j_1}, \ldots, x_{j_{n-r}}$ correspond to the free variables of $Ax = \vec{0}$. But by THE RANK THEOREM, $\dim \mathrm{Nul}(A) = n - \dim \mathrm{Col}(A) = n - r$. Thus $\{\vec{v}_1, \ldots, \vec{v}_{n-r}\}$ is a basis of $\mathrm{Nul}(A)$ by THEOREM 5.

Examples: Textbook P.240, P.241, P.264.

6 Rank & Matrix Inverse

Theorem 13 (The Invertible Matrix Theorem)**.** *Let A be an $n \times n$ matrix. The following statements are each equivalent to the statement that A is an invertible matrix:*
(1) The columns of A form a basis of $\mathbb{R}^n$;
(2) $ColA = \mathbb{R}^n$;
(3) dim $ColA = \mathbb{R}^n$;
(4) $rankA = n$;
(5) $NulA = \{\vec{0}\}$;
(6) dim $NulA = 0$;

Flora, *by Titian*